A MODEL OF THE STEADY-STATE MOTION OF SUSPENSIONS

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The authors examine the steady-state one-dimensional motions of suspensions whose particles have a density equal to that of the corresponding dispersion medium. As a whole, the mechanical behavior of such suspensions is described by equations of motion that coincide in form with the Navier-Stokes equations for a certain incompressible fluid whose viscosity is a known function of the particle concentration in the suspensions. To close these equations, the authors postulate a principle of minimum energy dissipation for steady-state motion, which plays the part of an equation of state for the suspension. This new equation permits the determination of the spatial distribution in the concentration of solids. Exact solutions are presented for certain variational problems associated with the Poiseuille flow of a fluid of this kind in circular tubes and Couette flows between concentric cylinders and parallel planes. It is shown that in most cases separation of the suspension takes place.

In [1], equations based on a model in which the phases are treated as interpenetrating interacting continua were proposed as a means of describing the motion of two-phase disperse systems. However, these equations are not sufficient for a complete description of the motion, since there is no analog of the equation of state of the disperse system as a whole that would make it possible to determine the behavior of the system concentration as a function of space and time.

It is easy to show, for example, that, even in the case of simple steady-state motion in a vertical circular tube, these equations have an arbitrary number of different solutions corresponding to various spatial distributions of the disperse phase (any physically admissible distribution has its own particular solution). Generally speaking, to eliminate this shortcoming of the theory and close the system of equations it is necessary to consider an additional equation of the quasidiffusional type. This equation must describe the variation of particle concentration in the system, with allowance for the dependence of the diffusion processes on the phase velocity gradients, the stresses in the flow, etc. It should evidently be some generalization of the Langevin or Fokker-Planck equations. Obviously, its formulation and the evaluation of the kinetic coefficients are very serious independent problems, whose solution involves almost insuperable difficulties. However, if we confine ourselves exclusively to steady-state motion, then by analogy with the known results of the thermodynamics of irreversible processes, we can introduce the heuristic principle that the disperse-phase concentration distribution in the flow is established in such a way that the dissipation of flow energy is at a minimum. This principle of minimum dissipation corresponds to the principle of maximum entropy increase from the thermodynamics of irreversible processes, although the latter has also not been rigorously proved for nonequilibrium mechanical systems.

It may be assumed that in a two-phase flow there are certain small random motions and associated forces caused, for example, by the wakes formed behind moving particles, by the interaction of the particles as well as by every conceivable fluctuation or perturbation of the phase velocities, disperse-phase concentration, and pressure. Under the action of such forces, the individual particles move into certain "equilibrium" positions corresponding to a given steady-state motion (and satisfying the formulated principle). The random force field and the process of establishment of the stationary motion may evidently be described and the principle of minimum energy dissipation justified only within the framework of the corresponding statistical theory. We note, however, that the existence of fluctuating forces and regions of true steady-state motion was experimentally demonstrated. (For example, in [2] the motion of individual particles in a viscous fluid was investigated). Without a detailed examination of the statistics of the above-mentioned forces, it is also impossible to determine the limits of applicability of the theory developed below. However, from general considerations it follows that at very low velocities these forces are very small, so that the steady-state regime may not be achieved in time for it to play an important part. On the other hand, at high velocities the random forces due to the fluctuations may be very intense so that the true distribution of the solid phase in the flow will vary considerably from the theoretical distribution corresponding to minimum energy dissipation.

For simplicity, we will examine the steady-state motion of suspensions of solid particles, whose density is equal to the density of the viscous medium. In this case, it is possible, with sufficient accuracy, to assume that the local values of the phase velocities coincide everywhere except for thin layers at the confining walls. Passing from the equations of [1] to the equivalent equations of two-fluid hydrodynamics of a disperse system [3], we demonstrate that the equations of motion for the suspension as a whole coincide with the Navier-Stokes equations for a viscous fluid with a viscosity which is a known function of the particle concentration at the walls [3]. Near the walls, the local values of the particle and fluid-phase valocities may not coincide and, moreover, the usual no-slip condition may not be satisfied. However, if the radius R of the tube is much greater than the radius a of the particles it can be shown that, within a/R, the boundary condition of no slip at the walls is satisfied. We also note that because of the practice of averaging over volumes whose linear dimensions considerably exceed a, all the results obtained below relate to the dimensions $l \gg a$.

§1. General formation of the problem of motion of suspensions. As usual, we can supplement the system of Navier-Stokes equations with the continuity equation of the fluid and the condition of conservation of the particle flow. Thus, we obtain the closed system in which

$$\frac{\partial p}{\partial x_i} = \frac{\partial}{\partial x_j}\left[\mu(\rho)\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right] - dg\delta_{i1};$$

$$\frac{\partial u_i}{\partial x_i} = 0; \qquad \frac{\partial(\rho u_i)}{\partial x_i} = 0; \qquad u_i|_\Gamma = U_i;$$

$$\min D = \min \frac{1}{2}\int_V \mu(\rho)\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)dV. \quad (1.1)$$

Here, d is the density and $\mu(\rho)$ is the viscosity of the fluid; ρ is the volume concentration of solids and U_i is the velocity of the solid surface Γ.

The effective viscosity $\mu(\rho)$ is, generally speaking, a concave function of ρ. For simplicity, we assume that as $\rho \to \rho_*$, where ρ_* is the concentration corresponding to dense packing of the particles, the quantity $\mu(\rho)$ tends to infinity; as $\rho \to 0$ we have $\mu \to \mu_0$, where μ_0 is the viscosity of the homogeneous disperse medium. It is convenient to pass from ρ to the normalized concentration ρ' using the definition $\rho = \rho'\rho_*$. Henceforth ρ will be understood to represent the normalized concentration ρ'. We note that it satisfies the natural inequalities

$$0 \leqslant \rho \leqslant 1. \quad (1.2)$$

In the specific calculations that follow we use an approximate relation for $\mu(\rho)$ satisfying the limiting conditions formulated above:

$$\mu(\rho) = \mu_0(1-\rho)^{-n} \qquad (n>1). \tag{1.3}$$

The variational problem (1.1)–(1.2) is a very complicated variant of a problem in optimum-control theory. Accordingly, in a number of cases it is possible to employ methods characteristic of that theory, i.e., either use the maximization principle [4] or reduce it to the functional equations of dynamic programming to obtain a numerical solution of the problem [5]. However, the maximization principle is ineffective in solving the problems considered in this paper, so it is preferable to make a qualitative investigation of the corresponding functionals in combination with the method of extremals.

§2. Flow in a circular tube. We will consider the axissymmetric steady-state flow of a suspension in an infinite circular tube of radius R. Equations (1.1) and the integrated conditions of conservation of the suspension (or fictitious fluid) and particle flows have the form

$$\frac{1}{r}\frac{d}{dr}\left(r\mu(\rho)\frac{du}{dr}\right) = \frac{\partial p}{\partial x} = -P;$$

$$U = \frac{2}{R^2}\int_0^R ru(r)\,dr; \qquad U_\rho = \frac{2\rho_*}{R^2}\int_0^R r\rho(r)u(r)\,dr. \tag{2.1}$$

Here, U and U_ρ are the mean flow velocities of the suspension and the particles, respectively. The expression for the energy dissipated per unit length of tube takes the form

$$D = \pi\int_0^R r\mu(\rho)\left(\frac{du}{dr}\right)^2 dr.$$

We introduce new dimensionless variables and the mean concentration over the section $\langle\rho\rangle$:

$$x = \left(\frac{r}{R}\right)^2; \qquad y = \frac{2}{R^2}\int_0^r r\rho(r)\,dr = \int_0^x \rho(\xi)\,d\xi;$$

$$y' = \rho(x); \quad y(0) = 0; \quad \langle\rho\rangle = \frac{2}{R^2}\int_0^R r\rho(r)\,dr. \tag{2.2}$$

Integrating the first of Eqs. (2.1) with allowance for the no-slip condition, substituting the results into the expressions for U, U_ρ, D and $\langle\rho\rangle$, and using expression (1.3) for $\mu(\rho)$, we obtain

$$D = \frac{P^2R^4}{16\mu_0}\int_0^1 x(1-y')^n\,dx = \frac{P^2R^4}{16\mu_0}J(y)$$

$$U = \frac{PR^2}{4\mu_0}\int_0^1 x(1-y')^n\,dx = \frac{PR^2}{4\mu_0}J(y)$$

$$U_\rho = \frac{PR^2\rho_*}{4\mu_0}\int_0^1 y(1-y')^n\,dx = \frac{PR^2\rho_*}{4\mu_0}G(y)$$

$$\langle\rho\rangle = \int_0^1 y'dx = y(1) = L(y), \quad y(0) = 0. \tag{2.3}$$

The velocity distribution over the cross section of the tube is described by the expression

$$u(x) = \frac{PR^2}{4\mu_0}\int_x^1 (1-y')^n\,dx. \tag{2.4}$$

Thus, to determine $\rho(x)$ it is necessary to solve a certain variational problem with the imposed isoperimetric condition and constraint (1.2). We will seek the solution $\rho(x)$ in the class of piecewise continuous functions $\rho(x) \equiv y'(x) \in C'(0,1)$, assuming, to be specific, that at the points of discontinuity $\rho \equiv y'$ is continuous on the left.

The motion of the suspension depends to a great extent on the nature of the given external conditions. For different external conditions and hence for different practical realizations of the motion the specific formulations of the variational problem will also be different. Below we consider three simple problems of the motion of a suspension:

1) flow defined by specifying the pressure gradient P and mean particle concentration $\langle\rho\rangle$;

2) flow defined by specifying $\langle\rho\rangle$ and the flow rate of the suspension U;

3) flow at specified suspension and particle flow rates, U and U_ρ.

All these problems can be realized, for example, by carrying out a corresponding experiment with a long capillary viscometer. Other types of flow, for example flow with given P and U_ρ, are also possible.

For given $\langle\rho\rangle$ and P, minimum D(y) corresponds to a minimization of the functional J(y). For a flow with given $\langle\rho\rangle$ and U, from (2.3) we have

$$P = 4\mu_0 UR^{-2}[J(y)]^{-1}, \quad D = \mu_0 U^2[J(y)]^{-1}.$$

Hence it follows that in the latter case min D(y) corresponds to max J(y) if $L(y) = \langle\rho\rangle$. Accordingly, it is convenient to examine the first two variational problems together.

§3. Solution of variational problems corresponding to a given mean particle concentration. We introduce the new variable $z = 1 - y' = 1 - \rho$. It is easy to see that $0 \le z \le 1$. The problems can then be formulated as follows: to find in the class of functions $z' \in C'(0, 1)$ functions that minimize and maximize the functional

$$J_0(z) = \int_0^1 xz^n\,dx \qquad (n>1), \tag{3.1}$$

with the condition

$$L_0(z) = \int_0^1 z\,dx = \langle z\rangle \quad (0 < \langle z\rangle < 1), \tag{3.2}$$

and the restraint

$$0 \leqslant z \leqslant 1. \tag{3.3}$$

The principal difficulty is that the unknown extremals do not, generally speaking, satisfy the Euler equation, which involves the possibility of an arbitrary number of discontinuities of the extremals on [0, 1]

Also, the possibility that the extremals will reach the boundaries $z = 0$ or $z = 1$ of the permissible region of variation for $z(x)$ must be anticipated. The fundamental fact that follows from a qualitative examination of the functionals is the monotonicity of the extremals. The proof is based on a lemma.

Lemma 3.1. On any interval Δ of nonzero measure ($\Delta \subseteq (0, 1)$) on which it is continuous, the minimum (maximum) $z_0 \in C'(0,1)$ representing the solution of variational problem (3.1)–(3.3) does not increase (decrease).

Proof. We assume the opposite, namely, that on the interval of continuity $\Delta(x_1, x_2)$ where $(0 \le x_1 < x_2 \le 1)$ the minimum $z_0(x)$ does not decrease. We construct the function

$$z_*(x) = \begin{cases} z_0(x_1 + x_2 - x), & x \in \Delta \\ z_0(x), & x \overline{\in} \Delta \end{cases}$$

representing the reflection of $z_0(x)$ on the interval Δ about the straight line $x = (x_1 + x_2)/2$. The function $z_*(x)$ satisfies constraint (3.3) and also condition (3.2), since the areas under the z_0 and z_* curves are the same. By virtue of the convolution property,

$$\int_{x_1}^{x_2} f(x)\,\varphi(x_1 + x_2 - x)\,dx = \int_{x_1}^{x_2} f(x_1 + x_2 - x)\,\varphi(x)\,dx;$$

after computations we obtain

$$J_0(z_*) - J_0(z_0) = $$
$$= 2\int_0^{1/2(x_2 - x_1)} \xi\left[z_0^n\left(\frac{x_1 + x_2}{2} - \xi\right) - z_0^n\left(\xi + \frac{x_1 + x_2}{2}\right)\right]d\xi \leqslant 0.$$

The latter conditional inequality follows from the fact that $z_0(x)$ is nondecreasing on Δ, the equality being realized only at $z_0(x) \equiv \equiv \text{const}$ $(x \in \Delta)$. We note that it occurs for arbitrary $n > 0$ in (1.3). The proof for the maximum, which by contradiction is assumed to be nonincreasing, is analogous. The contradiction obtained also proves the lemma.

To prove the monotonicity of the extremals on the interval [0, 1], it is necessary to show that the minimum (maximum) cannot increase (decrease) abruptly. A discontinuity of the piecewise-monotonic function $z(x) \in C'$ is regular if it preserves the monotonicity of $z(x)$; otherwise it will be called irregular. We have the following theorem.

Theorem 3.1. The minimum (maximum) for the problem (3.1)–(3.3) does not increase (decrease) on the interval [0, 1].

Proof. We will prove this theorem for the minimum; the proof for the maximum is analogous. Assuming the opposite, namely, that the minimum $z_0(x)$ has N irregular discontinuities at points $0 < x_0 < \ldots \ldots < x_{N-1}$ and examining z_0 on $[0, x_1]$, we let $z_0^- = z_0(x_0 - 0)$, $z_0^+ = = z_0(x_0 + 0)$, where, by assumption $z_0^+ > z_0^-$. We also denote $z_0(x)$ on the left of x_0 by $\varphi_1(x)$ and, on the right of z_0 (i.e., on the interval (x_0, x_1)) by $\varphi_2(x)$ and isolate in the neighborhood of x_0 a small interval $\Delta(x_\alpha, x_\beta) \subseteq [0, x_1]$ such that $\varphi_1(x_\alpha) \le \varphi_2(x_\beta)$. The interval Δ exists in view of the piecewise continuity and piecewise monotonicity (Lemma 3.1) of $z_0(x)$. On the interval $\Delta(x_\alpha, x_\beta)$ we construct the new function $z_*(x)$ (represented by the line C'D'A'B') from the function z_0 (represented by the line ABCD (Fig. 1)). For this purpose we move the piece CD horizontally to the left, parallel to itself until the point C coincides with C' with abscissa x_α; piece AB is similarly shifted to the right until the point B coincides with B' with abscissa x_β. Conditions (3.2) and (3.3) are invariant under these parallel translations. The analytical representation of $z_*(x)$ has the form

$$z_*(x) = \begin{cases} z_0(x), & x \overline{\in} \Delta\,(x_\alpha, x_\beta) \\ \varphi_2(x - x_\alpha + x_0), & x_\alpha < x < x_\alpha + x_\beta - x_0 \\ \varphi_1(x - x_\beta + x_0), & x_\alpha + x_\beta - x_0 < x < x_\beta. \end{cases}$$

After a simple calculation, we obtain

$$J_0(z_*) - J_0(z_0) = (x_\beta - x_0)\int_{x_\alpha}^{x_0} \varphi_1^n(x)\,dx - (x_0 - x_\alpha)\int_{x_0}^{x_\beta} \varphi_2^n(x)\,dx.$$

In the second integral we introduce the change of variables

$$x = a\xi + b, \quad a = \frac{x_\beta - x_0}{x_0 - x_\alpha}, \quad b = x_0 - ax_\alpha.$$

which transforms the segment (x_0, x_β) into the segment (x_α, x_0). Then

$$(x_0 - x_\alpha)\int_{x_0}^{x_\beta} \varphi_2^n(x)\,dx = (x_\beta - x_0)\int_{x_\alpha}^{x_0} \varphi_2^n(ax + b)\,dx.$$

Since, from the construction of the interval Δ we have $\varphi_1 < \varphi_2$, we may therefore obtain

$$J_0(z_*) - J_0(z_0) = (x_\beta - x_0)\int_{x_\alpha}^{x_0} [\varphi_1^n(x) - \varphi_2^n(ax + b)]\,dx < 0.$$

If $x_\alpha = 0$ and $x_\beta = x_1$ (i.e., $z_0(0) \le z_0(x_1)$), the minimum cannot have an irregular discontinuity at the point x_0.

We will now consider the general case. Without loss of generality it is sufficient to consider the situation characterized by the inequalities

$$z_0(0) > z_0^+ > z_0^- > z_0(x_1). \tag{3.4}$$

In fact, let, for example, $z_0(0) < z_0^+$ and $z_0(x_1) < z_0^-$. Then the function $z_*(x)$, represented by the dashed line C'E'B'ED in Fig. 2a, will give a smaller value for the functional $J_0(z)$ than $z_0(x)$, represented by the dashed line ABCED conditions (3.2) and (3.3) must be invariant under the transition from z_0 to z_*). It is easy to see that for $z_*(x)$, which alone need be considered, inequalities (3.4) do, in fact, hold. The argument is quite analogous for the case in which

$$z_0^- < z_0(x_1), \quad z_0(0) > z_0^+$$

Thus, let $z_0(x)$, satisfying (34), have the form shown in Fig. 2b. We draw the straight line $z = z^0 = (z_0^+ + z_0^-)/2$ parallel to the axis of abscissas until it intersects the curves $z = \varphi_1(x)$ and $z = \varphi_2(x)$ at points A and D, respectively (Fig. 2b), with abscissas x_α and x_β. The function $z_1(x)$, which coincides with $z_0(x)$ everywhere outside the interval $\Delta(x_\alpha, x_\beta)$ and on Δ is represented by the dashed line AC'B'D, gives a lesser value of the functional $J_0(z)$ than $z_0(x)$. We note that the constructed function $z_1(x)$ has two irregular discontinuities at the points x_α and x_β on $[0, x_1]$, the value of these discontinuities being equal to half the original discontinuity. Going through a similar procedure with the irregular discontinuities of the function $z_1(x)$, we obtain a function $z_2(x)$ giving a lesser value of the functional with satisfaction of conditions (3.2) and (3.3). $z_2(x)$ has four irregular discon-

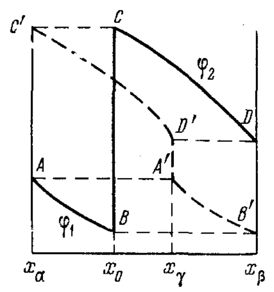

Fig. 1

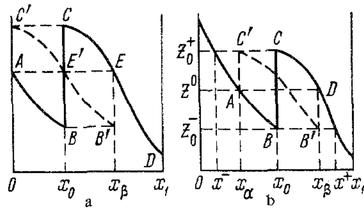

Fig. 2

tinuities, each of which is equal to one-fourth of the original discontinuity of the function $z_0(x)$. The abscissas of these discontinuities never coincide. It is easy to see, by induction, that the k-th function $z_k(x)$ has 2^k irregular discontinuities of magnitude $2^{-k}(z_0^+ - z_0^-)$. Continuing the process indefinitely, we obtain a function $z_n(x)$ having no discontinuities on the interval $[0, x_1]$ at which $z_n(x)$ increases and giving functional (3.1) a smaller value than $z_0(x)$ would give it. Repeating the proof for the interval $[0, x_2]$, we find that on this interval also the minimum cannot have irregular discontinuities. Continuing the argument by induction, we arrive at the same conclusion for the entire interval $[0, 1]$. This proves the theorem.

We note that in the proof only the fact that $n > 0$ was essential. This corresponds to an increase in the fluidity of the suspension with increase in z. Instead of the fluidity z^n in (3.1), which follows from expression (1.3) for the effective viscosity, an arbitrary increasing function $\psi(z)$, such that $\Psi(0) = 0$, might have figured in the proof.

A priori, two situations are possible: either the minimum (maximum) is confined to the boundary of the region taking the values zero or unity, or on a certain interval it enters the open region $0 < z < 1$. In the first case it can have only one discontinuity; in the second case, on the interval where $0 < z < 1$ it is described by the Euler equation. The Euler equation for functional (3.1) with condition (3.2) is

$$nxz^{n-1} = \lambda; \qquad z = \left(\frac{x_0}{x}\right)^{1/(n-1)}; \qquad x_0 = \frac{\lambda}{n}. \tag{3.5}$$

It follows from (3.5) that an extremal of the indicated type exists only at $n \neq 1$, while, as may easily be shown, for $n > 1$ function (3.5) gives functional (3.1) a local minimum and for $n < 1$ a local maximum. We will prove Lemma 3.2.

Lemma 3.2. If there exists an interval $\Delta(x_1, x_2)$ on which the extremal satisfies the condition $0 < z_0 < 1$, then on this interval $z_0(x)$ is continuous.

Proof. It is sufficient to consider only such Δ on which a single discontinuity of the extremal $z_0(x)$ exists. In view of the additivity of functionals (3.1) and (3.2), $z_0(x)$ is a solution of the isoperimetric problem of the extremum of $J_0(z_0, \Delta)$:

$$J_0(z_0, \Delta) = \int_{x_1}^{x_2} x z_0^n\, dx, \qquad \int_{x_1}^{x_2} z_0\, dx = \langle z_0\rangle(\Delta).$$

Introducing the new variable $z_0 = w' = dw/dx$ and considering as usual the functional

$$\int_{x_1}^{x_2} [x(w')^n - \lambda w']\, dx = \int_{x_1}^{x_2} F(x, w')\, dx,$$

we find the extremal $w(x)$ must satisfy the Euler equation

$$F_{w'} = nx(w')^{n-1} - \lambda = A = \text{const}, \tag{3.6}$$

and the Weierstrass-Erdmann conditions at the point of discontinuity $c \in \Delta$

$$[F_{w'}]_{c-0}^{c+0} = 0, \qquad [F - w'F_{w'}]_{c-0}^{c+0} = 0. \tag{3.7}$$

From (3.6) and (3.7) it follows that

$$[F_{w'}]_{c-0}^{c+0} = nc\{[w'(c+0)]^{n-1} - [w'(c-0)]^{n-1}\} = 0.$$

Hence there follows the continuity of $w' = z$ on Δ.

From the proved Lemma 3.2 it follows that on $[0,1]$ the extermal $z_0(x)$ can have not more than two discontinuties at the points $x^{(1)}$ and $x^{(2)}$, the inequalities $z_0(x^{(1)} - 0) = 1$ and $z_0(x^{(2)} + 0) = 0$, being satisfied for the minimum and the inequalities $z_0(x^{(1)} - 0) = 0$, $z_0(x^{(2)} + 0) = 1$ for the maximum. We now have Theorem 3.2.

Theorem 3.2. If n in (3.1) is greater than unity, the minimum $z_0(x)$ of the variational problem is unique and continuous and can be represented by the expressions

$$z_0(x) = \begin{cases} 1, & 0 \leqslant x \leqslant x_0 \\ (x_0/x)^{1/(n-1)}, & x_0 < x \leqslant 1, \end{cases} \tag{3.8}$$

where x_0 is the unique root of the transcendental equation

$$\langle z\rangle = \frac{n-1}{n-2} x_0^{1/(n-1)} - \frac{x_0}{n-2}, \tag{3.9}$$

while the maximum $z^0(x)$ is unique and given by the expressions

$$z^0(x) = \begin{cases} 0, & 0 \leqslant x \leqslant 1 - \langle z\rangle \\ 1, & 1 - \langle z\rangle < x \leqslant 1 \end{cases} \tag{3.10}$$

Proof. As before, we proceed by contradiction. In accordance with Lemma 3.2, it is possible, without loss of generality, to assume that the minimum is represented by the dashed line ABCDEF in Fig. 3, the segments AB and EF lying in ε-neighborhoods of the straight lines $z = 1$ and $z = 0$, respectively, while the curve CD is given by Eq. (3.5). We construct a new function such that conditions (3.2) and (3.3) do not change and such that this function coincides with the previous function at $x < x_\alpha$ and $x > x_2$ (dashed line AB'D'EF in Fig. 3). Using Lemma 3.2 and Eq. (3.5), it is easy to see that the curve B'D' is unique and also belongs to the class of (3.5), the role of x_0 in (3.5) being played by the quantity $x_\alpha, x_\alpha < x_1$. A direct calculation shows that AB'D'EF gives functional (3.1) a value smaller than does ABCDEF. Letting ε tend to zero, we obtain proof of the continuity of the minimum on the left-hand side. We then construct the function AB''F', which coincides with AB'D'EF in the region $x < x_0$ and does not affect the conditions (3.2) and (3.3). In the region $x > x_0$ this function also belongs

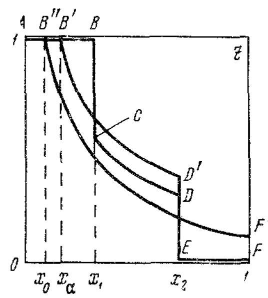

Fig. 3

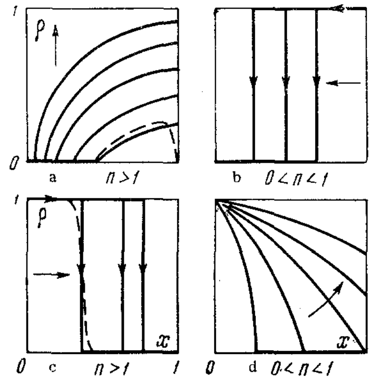

Fig. 4

to the class of (3.5), since there is always a corresponding value of x_0. Direct calculations show that it gives functional (3.1) a smaller value than AB'D'EF. Consequently, the minimum $z_0(x)$ is also continuous on the right-hand side and can be represented by expressions (3.8).

The quantity x_0 in (3.8) is found from the isoperimetric condition (3.2), which leads to Eq. (3.9). The right-hand side $f(x_0)$ of this equation increases monotonically on the interval [0, 1] from zero to unity, since

$$f'(x_0) = (x_0^{-(n-2)/(n-1)} - 1)\ (n-2)^{-1} > 0, \quad n > 1.$$

Hence it follows that if the condition $0 < \langle z \rangle < 1$ is satisfied there will always exist a unique root of Eq. (3.9), whence follows the uniqueness of the minimum for n > 1.

When n > 1, the second assertion of the theorem is obvious. In fact, if it is assumed that there is a certain interval $\Delta(x_1, x_2)$ on which the maximum z^0 satisfies the inequality $0 < z^0 < 1$, on a certain inner segment of that interval (where z^0 is continuous, and such a segment always exists since $z^0 \in C'$), the maximum must satisfy the Euler equation (3.5). However, for n > 1 the latter only has solutions corresponding to a minimum of functional (3.1). Hence it follows that $z^0(x)$ can only take values of zero or unity. Then, by Theorem 3.1 the maximum can be represented in the form of (3.10). This proves the theorem.

When 0 < n < 1 it can be shown in exactly the same way that the unique minimum can be represented in the form

$$z_0(x) = \begin{cases} 1, & 0 \leqslant x \leqslant 1 - \langle z \rangle \\ 0, & 1 - \langle z \rangle < x \leqslant 1 \end{cases}$$

while the unique maximum is given by the expressions

$$z^0(x) = \langle z \rangle \frac{2-n}{1-n} x^{1/(1-n)}, \qquad \langle z \rangle < \frac{1-n}{2-n};$$

$$z^0(x) = \begin{cases} (x/x_0)^{1/(1-n)}, & 0 \leqslant x \leqslant x_0 \\ 1, & x_0 < x \leqslant 1 \end{cases}, \qquad \langle z \rangle > \frac{1-n}{2-n}.$$

We now note that all the qualitative results (and the quantitative results relating to discontinuous distributions) correspond to an arbitrary fluidity function $\Psi(z)$, increasing with z ($\Psi(0) = 0$), that can be substituted for z^n in (3.1). Here, the case n > 1 corresponds to concave functions $\Psi(z)$, and the case n < 1 to convex $\Psi(z)$. It is easy to prove the possibility of generalizations of this kind by examining the proofs of the theorems.

On the basis of the results obtained, the qualitative form of the particle concentration distributions in the suspension $\rho(x)$ may be represented by the curves in Fig. 4 (a and b are the minima; c and d are the maxima); the arrows indicate the direction of displacement of the curves with increase in $\langle \rho \rangle$. Experiments indicate that the actual $\mu(\rho)$ are concave functions; accordingly, in actual cases we have qualitative distributions corresponding to n > 1 in Fig. 4.

We also note that, as indicated in §1, the results cannot be valid at distances from the wall on the order of the particle size, since lubricating layers that repel the particles develop at the walls. Similarly, the analytically obtained discontinuities of the distributions $\rho(x)$ actually correspond to a sharp change in $\rho(x)$ at distances on the order of a. Hence it follows that the actual distributions differ somewhat from those obtained above (see dashed lines in Fig. 4). With the formulas of §2, from the known functions $\rho(x)$ it is easy to obtain expressions for the integral characteristics of motion of the suspension and the velocity field for the flow. These formulas have been omitted for lack of space.

§4. Solution of the problem corresponding to given flow rates of the particles and the suspension as a whole. By eliminating the unknown pressure gradient P from (2.3), we arrive at the problem of finding the $y_0 \in C_1'$ that maximizes the functional

$$J(y) = \int_0^1 x(1 - y')^n\, dx, \qquad y(0) = 0 \tag{4.1}$$

with the isoperimetric condition

$$G(y) - KJ(y) = \int_0^1 (y - Kx)(1 - y')^n\, dx = 0,$$

$$K = \frac{U_\rho}{\rho_* U} < 1 \tag{4.2}$$

and the constraint

$$0 \leqslant y'(x) \leqslant 1. \tag{4.3}$$

It is easy to see that the maximum $y_0(x)$, representing the solution of problem (4.1)–(4.3), monotonically increases (does not decrease) and is located in a triangle bounded by the axis of abscissas (y = 0), the bisector of the first quadrant and the straight line x = 1. It is clear from (4.2) that the maximum $y_0(x)$ simultaneously maximizes the functional G(y) if the isoperimetric condition (4.2) is satisfied.

Let there exist a maximum y_0. Then, there also exists

$$\max y_0(x) = y_0(1) = \langle \rho \rangle \qquad (0 < \langle \rho \rangle < 1) \tag{4.4}$$

We will consider the following auxiliary problems.

Problem 1. To find max J(y) in the class with condition (4.4), constraint (4.3), and the condition y(0) = 0.

Problem 2. To find max G(Y) in the class $y \in C_1'$ with the same conditions and constraint.

Problem 1 was solved in the previous section; in the notation of this section, its solution has the form

$$y_1(x) = \begin{cases} x, & 0 \leqslant x \leqslant \langle \rho \rangle \\ 0, & \langle \rho \rangle < x \leqslant 1 \end{cases} \tag{4.5}$$

It is found that the solution of problem 2 coincides with $y_1(x)$ from (4.5). In exactly the same way as in §3, it is proved that for n > 1 (or generally for a concave viscosity characteristic) the solution of the Euler equation gives the functional G(y) a local minimum under the above-mentioned conditions. Hence it immediately follows that the unknown maximum $y_2(x)$ of problem 2 has a derivative equal either to zero or to unity, i.e., always moves along the boundary of region (4.3). It can be represented by the dashed line shown in Fig. 5. It is easy to see that of all such dashed lines the curve (4.5) (the dashed line OA'F in Fig. 5) gives the greatest value of the functional G(y). This follows from the fact that only the "plateau" regions, where $y' \equiv 0$, contribute to G(y), which is simply equal to the sum of the areas under these regions (Fig. 5).

From this there follows a corollary: for given $\langle\rho\rangle$, the minimum dissipation principle, when U is also given, is equivalent to the principle of maximum particle flow rate, when P is given. To prove this assertion it is sufficient to consider (2.3).

The value of $\langle\rho\rangle$ in (4.5) is found from condition (4.2):

$$\langle\rho\rangle = K(2-K)^{-1} \tag{4.6}$$

We note that if a suspension with given U and U_ρ is supplied to the tube inlet, K may be regarded as the mean inlet concentration. Then, in accordance with (4.6), in the region of steady-state flow, the mean concentration $\langle\rho\rangle$ may differ substantially from K, this difference being especially noticeable at small K.

We also note that if isoperimetric condition (4.2), which determines the relation between the particle flow and the flow of the suspension as a whole, and the pressure gradient P are given, the principle of minimum energy dissipation leads to the conclusion that there is no flow in the tube. Physically, this can easily be understood by considering a special experiment in which a suspension with given mean concentration K is admitted to a cylinder bounded by a piston under constant pressure. The piston forces the suspension into a capillary of much smaller diameter. A certain time after the process begins the particles in the capillary enter the densely packed state, after which motion ceases, and new suspension entering the cylinder only serves to raise the piston.

§5. Couette flow. We will consider Couette flow between concentric cylinders of radii R_1 and R_2 ($R_1 < R_2$). The system of equations corresponding to (2.1) then has the form

$$d\frac{v_\varphi^2}{r} = \frac{dp}{dr}; \qquad \frac{dp_{r\varphi}}{dr} + \frac{2p_{r\varphi}}{r} = 0;$$

$$p_{r\varphi} = \mu(\rho)\, r\frac{d}{dr}\left(\frac{v_\varphi}{r}\right); \qquad D = 2\pi\int_{R_1}^{R_2} r p_{r\varphi}\dot{\gamma}_{r\varphi}\,dr\,.$$

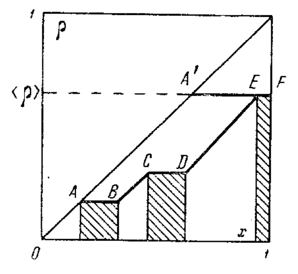

Fig. 5

Introducing the dimensionless variable $x = (r/R_2)^2$ and using Eq. (1.3) in the integration, we obtain a system of functionals characterizing the motion:

$$D = \frac{\pi\tau_0^2 R_2^2}{\mu_0}\int_k^1 z^n\frac{dx}{x^2} = \frac{\pi\tau_0^2R_2^2}{\mu_0}M(z), \; k = \left(\frac{R_1}{R_2}\right)^2;$$

$$V = \frac{R_2\tau_0}{\mu_0}\int_0^1 z^n\frac{dx}{x^2} = \frac{R_2\tau_0}{\mu_0}M(z), \; z = 1-\rho;$$

$$\langle z\rangle = \frac{1}{1-k}\int_k^1 z\,dx = \frac{1}{1-k}L(z). \tag{5.1}$$

Here, τ_0 is the stress at the outer cylinder, D is the dissipated energy, and V is the linear velocity at the outer cylinder. The velocity of the inner cylinder is assumed to be equal to zero, which, as may easily be seen, does not restrict the generality of the analysis. Thus, for example, if the inner cylinder rotates and the outer cylinder is stationary, the second functional in (5.1) does not change, while the first and third are generally constant. As in §3, we arrive at a variational problem.

In the class $z \in C'$ to find the function $z_0(x)$ that minimizes (maximizes) the functional

$$M(z) = \int_k^1 z^n\frac{dx}{x^2} \tag{5.2}$$

with the condition

$$N(z) = \frac{1}{1-k}\int_k^1 z\,dx = \langle z\rangle \quad (0 < \langle z\rangle < 1) \tag{5.3}$$

and the constraint

$$0 \leqslant z \leqslant 1\,. \tag{5.4}$$

We note that the problem of min M(z) corresponds to determination of the concentration field when the mean concentration in the suspension and the stress at the outer cylinder are given. The problem of max M(z) corresponds to determination of the concentration field when the mean concentration of the suspension and the velocity at the outer cylinder are given.

As in §3, the behavior of the extremals is determined from a qualitative investigation of the variational problem (5.2)–(5.4) using the Euler equation. The qualitative behavior of the extremals is defined by two assertions analogous to Lemma 3.1 and Theorem 3.1.

Lemma 5.1. The function $z_0 \in C'(k, 1)$ minimizing (maximizing) functional (5.2) with condition (5.3) and constraint (5.4) does not decrease (increase) on any interval of continuity Δ of nonzero measure.

Theorem 5.1. The function $z_0 \in C'(k, 1)$ that minimizes (maximizes) functional (5.2) with condition (5.3) and constraint (5.4) does not decrease (increase) on the entire interval [k, 1].

These assertions are proved by the same methods as employed in proving Lemma 3.1 and Theorem 3.1. The "somewhat inverted nature" of these assertions as compared with the assertions of §3 is attributable to the difference between the optimized functionals J(z) and M(z).

Then, in the same way as in §3, it is shown that, on its intervals of continuity, the minimum is described by the corresponding Euler equation, whose solution can have corner points only at the boundaries z = 0 and z = 1 of the region of variation of z_0. The maximum, however, can only take the values z = 0 or z = 1. Since the maximum is monotonic (nonincreasing), it has the form

$$z_0(x) = \begin{cases} 1, & k \leqslant x \leqslant \alpha \\ 0, & \alpha < x \leqslant 1 \end{cases} \quad (\alpha = k + (1-k)\langle z \rangle). \tag{5.5}$$

Here, the constant α is found from the isoperimetric condition.

The family of intervals of the minima satisfying the Euler equation is represented by the expressions

$$z_0(x) = Cx^{2/(n-1)} \qquad (C > 0);$$
$$C = \frac{\langle z \rangle (1-k)(n+1)}{[1 - k^{(n+1)/(n-1)}](n-1)} \qquad (n > 1). \tag{5.6}$$

From (5.6) it is easy to see that for $1 < n < 3$ the function $z_0(x)$ is concave, and for $n > 3$ it is convex. The problem of the minimum falling on the boundary z = 1 is solved by investigating the behavior of $z(1) = C(\langle z \rangle, k, n)$, where

$$\frac{n+1}{n-1} = \beta > 1; \quad C(\langle z \rangle, k, n) = \langle z \rangle \beta \frac{1-k}{1-k^{\beta}}.$$

For fixed $\langle z \rangle$, β we have

$$C|_{k=0} = \langle z \rangle \beta; \quad C|_{k=1} = \langle z \rangle; \quad \frac{dC}{dk} < 0; \quad \left.\frac{dC}{dk}\right|_{k=1} = 0. \tag{5.7}$$

The behavior of $C(\langle z \rangle, k, \beta)$ as a function of k is presented in Fig. 6. If $\langle z \rangle \beta > 1$, there exists a k_0 such that for $k < k_0$ we have $C > 1$. If $\langle z \rangle \beta \leq 1$, $C < 1$ for all possible k. In the first case, with $k < k_0$ (sufficiently wide gaps) at a certain $x = x_0$ the minimum reaches the boundary z = 1 and then travels along the boundary, while for $k > k_0$ the minimum does not reach the boundary. Thus, the behavior of the minima (i.e., the solutions of the problem for given τ_0 and $\langle \rho \rangle$) and the maxima (solutions for given V and $\langle \rho \rangle$ is analogous to the behavior of the curves in Fig. 7a, b, respectively. When the gaps are very narrow ($k \to 1$), it follows from (5.7) that, for given

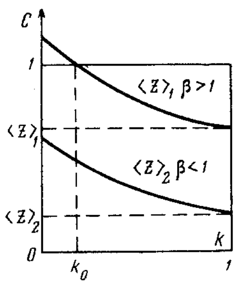

Fig. 6

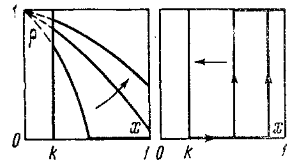

Fig. 7

τ_0 and $\langle \rho \rangle$, the suspension concentration distribution tends to become uniform.

The flow and concentration distribution in plane Couette flow are easily obtained from (5.5) and (5.6) by passing to the limit as $R_1 \to \infty$ with $R_2 - R_1 = h =$ = const and $r - R_1 = y$ = const. For the minimum, from (5.6) we have

$$z_1(y) = \lim z_0(x) = \langle z \rangle, \; R_1 \to \infty, \; r - R_1 = \text{const}. \tag{5.8}$$

For the maximum, from (5.5), we obtain

$$z_2(y) = \lim z_0(x) = \begin{cases} 1, & 0 \leqslant y \leqslant \langle z \rangle \\ 0, & \langle z \rangle < y \leqslant 1 \end{cases}, \; R_1 \to \infty, \; r - R_1 = \text{const}. \tag{5.9}$$

It should be noted that a direct investigation of the maximum in plane Couette flow shows that there is a continuum of functions $\{z_0(y)\} \in L(0, 1)$ that maximize the corresponding functional under conditions of the type of (5.3) and (5.4). These functions are represented by the expressions

$$z_0(y) = \begin{cases} 0, & y \in E, \; \text{mes}\, E = 1 - \langle z \rangle \\ 1, & y \in [0, 1] / E = E', \; \text{mes}\, E' = \langle z \rangle \end{cases} \tag{5.10}$$

The proof follows from the fact that the maximum $z \in C'$ can take values of zero or unity (this follows from an investigation of the Euler equation for the given problem) and also from the invariance of functional (5.2) for k = 0 with respect to translation.

Then, since $z_0 \in L$ is the limit in the mean of the step functions $z_k \in C'$ having values of either zero or unity while there is no analog of Lemma 5.1 and Theorem 5.1 for plane Couette flow, we obtain the proof of Eq. (5.10).

§6. Discussion. The results obtained show that, rheologically, a suspension cannot be treated as an ordinary fluid. In particular, it does not possess such a unique structural constant as viscosity, since, as shown above, different viscometric experiments may lead to rather different viscosity relations. This observation (noninvariance of the rheological curves with respect to the type of viscometric experiment) has been repeatedly noted in the literature. The separation effect is also important in suspension flows. It should be noted that there has not yet been any direct experimental confirmation of the results obtained in the present research. Nonetheless, many experimental data indirectly indicate the existence of effects of this kind: the wall effect noted by various authors, the nonuniform distribution of erythrocyte concentration in blood plasma moving through the vessels, etc. The authors are also familiar with the experiments of G. V. Vinogradov with equal-density suspensions of soaps in various organic liquids using a constant-pressure capillary viscometer. In these experiments, a region primarily occupied by liquid phase was observed at the center of the capillary, while the particles were driven toward the capillary walls.

We note that the model employed should be useful for investigating the motions of nonequal-density suspensions (slurries, etc.), which play a very important part in a number of areas of technology (the oil industry, hydraulic engineering, chemical engineering, etc.).

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REFERENCES

1. G. I. Barenblatt, "Motion of suspended particles in a turbulent flow," PMM, vol. 17, no. 3, p. 261, 1953.

2. R. C. Jeffrey and J. R. A. Pearson, "Particle motion in laminar vertical tube flow," J. Fluid Mech., vol. 22, no. 4, p. 721, 1965.

3. Yu. A. Buevich, "Two-fluid hydrodynamics of a fluidized bed," Izv. AN SSSR, MZhG [Fluid Dynamics], no. 4, 1966.

4. L. S. Pontryagin, V. G. Boltyanskii, R. K. Gamkrelidze and E. F. Mishchenko, Mathematical Theory of Optimum Processes [in Russian], Fizmatgiz, 1961.

5. R. Bellman, Dynamic Programming [Russian translation], IL, 1960.

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